

Comparative Study of Damage Spreading in the Ising Model Using Heat-Bath, Glauber, and Metropolis Dynamics

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We study the time evolution of two configurations of the Ising model submitted to heat-bath (HB), Glauber (G), and two types of Metropolis (M and \bar{M}) dynamics, analyzing the damage spreading on a square lattice. We find that the damages produced by the dynamics G and M are greater than those resulting from HB and \bar{M} dynamics. We also observe that, only for zero magnetic field, the damages of the dynamics G and M seem to be numerically equivalent.

KEY WORDS:

1. INTRODUCTION

Statistical models of Boolean variables ($\sigma_i = 0, 1$), on the sites of a regular lattice, are used to describe the properties of a great variety of physical systems. By choosing appropriate expressions for the interactions between sites of the lattice and for the rules which determine the dynamic evolution of each variable, these models can describe magnetic and lattice-gas systems (Ising models⁽¹⁾), cellular automata,⁽²⁾ associated biological systems,⁽³⁾ and many others.

In recent decades, with the advance of modern computational techniques,⁽⁴⁾ the numerical and graphical study of these systems has been possible, and many microscopic aspects, such as the spreading of small perturbations or the stability of a determined configuration, turned out to be of intrinsic interest.

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The damage spreading problem consists in investigating the time evolution of two configurations $\{\sigma^A\}$ and $\{\sigma^B\}$ of the model which evolve by using the same dynamics and the same sequence of random numbers, and calculating its Hamming distance (or damage), defined by

$$D(t) = \frac{1}{2N} \sum_{i=1}^N |\sigma_i^A(t) - \sigma_i^B(t)| \quad (1)$$

For the particular case of the Ising model, the damage spreading has been investigated numerically by imposing different initial conditions and dynamical rules (heat bath,⁽⁴⁻⁶⁾ Glauber,⁽⁷⁻¹⁰⁾ and Q2R⁽⁷⁾), as well as by employing analytical methods^(11,12) which revealed the relation between the damage and static thermodynamic properties, such as the pair correlation function and the magnetization.

In this paper, we make a comparative study between several dynamics (heat bath, Glauber, and Metropolis). To do this, we analyze the damage spreading between two configurations $\{\sigma^A\}$ and $\{\sigma^B\}$ of the Ising model submitted to the same dynamics; this means that the site i in each of the two configurations at a given time t evolves at time $t+1$ to a new state which is determined by the same functional of the local fields $h_i^A(t)$ and $h_i^B(t)$. Moreover, we use the same random number to update corresponding spins in the two configurations.

In Section 2 we perform an analytical comparison between the heat-bath and Glauber dynamics; in Section 3 we analyze two types of Metropolis dynamics (M and \tilde{M}); in Section 4 we present the results of a numerical study of the damage spreading resulting from these four dynamics on a square lattice; finally, we conclude in Section 5.

2. THE HEAT-BATH AND GLAUBER DYNAMICS

At a given time t each site i of the Ising model has a local field $h_i(t) = \sum_j K_{ij} [2\sigma_j(t) - 1] + H$, where $K_{ij} \equiv J_{ij}/k_B T$ and $H = h/k_B T$ are the (dimensionless) first-neighboring coupling constant and magnetic field, respectively, and $\sigma_j(t) = 0, 1$ is a Boolean variable. We define an associated probability $p_i(t)$ by

$$p_i(t) = [1 + e^{-2h_i(t)}]^{-1} \quad (2)$$

The heat-bath dynamics selects, at a given time t on site i a random number $0 \leq X_i(t) \leq 1$, and determines the new state of the variable σ_i at a time $t+1$, using the rule

$$\sigma_i(t+1) = \begin{cases} 1 & \text{if } X_i(t) \leq p_i(t) \\ 0 & \text{if } X_i(t) > p_i(t) \end{cases} \quad (3)$$

If we denote by $P_i^{(\sigma_i^A, \sigma_i^B)}(t+1)$ the probabilities that, at time $t+1$, the variables σ_i^A and σ_i^B take a given value in the configurations $\{\sigma^A\}$ and $\{\sigma^B\}$, respectively, we can see from Eq. (3) that

$$\begin{aligned} P_i^{1,1}(t+1) &= \min(p_i^A(t), p_i^B(t)) \\ P_i^{0,0}(t+1) &= \min(1 - p_i^A(t), 1 - p_i^B(t)) \\ P_i^{1,0}(t+1) &= \max(0, p_i^A(t) - p_i^B(t)) \\ P_i^{0,1}(t+1) &= \max(0, p_i^B(t) - p_i^A(t)) \end{aligned}$$

Consequently, the probability $P_i(t+1)$ that the site i has a damage ($\sigma_i^A(t+1) \neq \sigma_i^B(t+1)$) is given by $P_i(t+1) = P_i^{1,0}(t+1) + P_i^{0,1}(t+1)$ and one has

$$P_i(t+1) \equiv P_i(\text{HB}) = |p_i^A(t) - p_i^B(t)| \tag{4}$$

In the Glauber dynamics the probability distribution which determines the value of $\sigma_i(t+1)$ depends also on the values of $\sigma_i(t)$. This dynamics is given by

$$\sigma_i(t+1) = \begin{cases} 1 & \text{if } X_i(t) \leq p_i(t) \\ 0 & \text{if } X_i(t) > p_i(t) \end{cases} \quad \text{when } \sigma_i(t) = 0 \tag{5}$$

and

$$\sigma_i(t+1) = \begin{cases} 0 & \text{if } X_i(t) \leq 1 - p_i(t) \\ 1 & \text{if } X_i(t) > 1 - p_i(t) \end{cases} \quad \text{when } \sigma_i(t) = 1 \tag{6}$$

To obtain the probability for damage on site i at a time $t+1$, ($P_i(t+1) \equiv P_i(\text{G})$), we follow the procedure of the HB case. But in this case we have to analyze different possibilities: For $\sigma_i^A(t) = \sigma_i^B(t)$

$$P_i(\text{G}) = |p_i^A(t) - p_i^B(t)| \tag{7}$$

For $\sigma_i^A(t) \neq \sigma_i^B(t)$, there are two cases to examine: (a) If $1 - p_i^A(t) \leq p_i^B(t)$,

$$P_i(\text{G}) = [1 - p_i^A(t)] + [1 - p_i^B(t)] \tag{8}$$

(b) If $1 - p_i^A(t) \geq p_i^B(t)$,

$$P_i(\text{G}) = p_i^A(t) + p_i^B(t) \tag{9}$$

We notice that from Eqs. (7)–(9) it follows that at $T = \infty$ the Glauber dynamics keeps the initial damage constant through the time evolution.

From (4) and (7)–(9) we can see that, for the same initial conditions at time t ,

$$P_i(\text{G}) \geq P_i(\text{HB}) \quad (10)$$

By remembering that the probability for total damage $D(t)$ is obtained by summing $P_i(t)$ over all sites of the lattice, we note that the Glauber dynamics enhances the damage as compared to the heat bath. Since this is true for an iteration from time t to time $t + 1$, it is true for all times.

3. THE METROPOLIS DYNAMICS

We have also examined the behavior of the damage spreading under the effect of two types of Metropolis dynamics (M and $\tilde{\text{M}}$). To describe them, it is useful to define two associated probabilities $p'_i(t)$ and $p''_i(t)$ given by

$$p'_i(t) = \min(1, e^{2h_i(t)}) \quad (11)$$

and

$$p''_i(t) = \min(1, e^{-2h_i(t)}) \quad (12)$$

The usual Metropolis dynamics (M) is defined by determining $\sigma_i(t + 1)$ through

$$\sigma_i(t + 1) = \begin{cases} 1 & \text{if } X_i(t) \leq p'_i(t) \\ 0 & \text{if } X_i(t) > p'_i(t) \end{cases} \quad \text{when } \sigma_i(t) = 0 \quad (13)$$

and

$$\sigma_i(t + 1) = \begin{cases} 0 & \text{if } X_i(t) \leq p''_i(t) \\ 1 & \text{if } X_i(t) > p''_i(t) \end{cases} \quad \text{when } \sigma_i(t) = 1 \quad (14)$$

To define the other Metropolis dynamics ($\tilde{\text{M}}$), we choose the value of $\sigma_i(t + 1)$ by the rule

$$\sigma_i(t + 1) = \begin{cases} 1 & \text{if } X_i(t) \leq \tilde{p}_i(t) \\ 0 & \text{if } X_i(t) > \tilde{p}_i(t) \end{cases} \quad (15)$$

where

$$\tilde{p}_i(t) = \begin{cases} p'_i(t) & \text{if } \sigma_i(t) = 0 \\ 1 - p''_i(t) & \text{if } \sigma_i(t) = 1 \end{cases} \quad (16)$$

By performing a similar analysis as was made in Section 2 and imposing the same initial condition at time t , we can prove that the respective probabilities for a damage on site i at a time $t + 1$ satisfy the relation

$$P_i(M) \geq P_i(\tilde{M}) \tag{17}$$

4. A NUMERICAL STUDY OF DAMAGE SPREADING

We have not been able to find an analytic relation like Eqs. (10) and (17) between the probabilities $P_i(M)$ and $P_i(\tilde{M})$ and the corresponding probabilities $P_i(G)$ and $P_i(HB)$.

We have, however, performed a numerical calculation of the damage spreading on a square lattice ($L = 40$) for the Ising model at temperature T , submitted to the different dynamics discussed previously.

To do this, we start with a thermalized configuration (A) of the model. At $t = 0$, we create another configuration (B), and we follow the time evolution of the configurations submitted to the same dynamics. After a long transient time ($t_1 \approx 1600$ steps per site) we take the time average of the damage over a long time of evolution ($t_2 \approx 6400$ steps per site), and take finally an average over several samples (≈ 40); only samples where the damage is not zero are considered.

For the initial conditions $\sigma_i^A(0) = \sigma_i^B(0)$ except the central site, on which $\sigma_0^A(0) = 1 - \sigma_0^B(0)$, that is, for initial damage $D(0) = 1/N$, we have found that:

- (a) The average damage $\bar{D}(HB) = \bar{D}(\tilde{M}) \approx 0$ for $T \geq 0$.

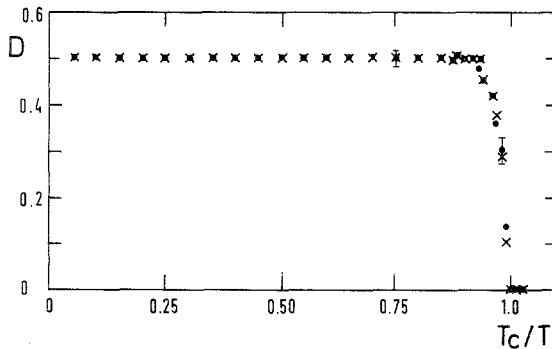


Fig. 1. Damage at zero magnetic field for (●) Glauber and (×) Metropolis M dynamics for a 40×40 square lattice. We measured the average over 40 samples (where the damage is not zero), and in each of them the damage is averaged over 6400 time steps per site. Initial condition $D(0) = 1/N$.

(b) At zero magnetic field, the average damages $\bar{D}(G)$ and $\bar{D}(M)$ (Fig. 1) are indistinguishable, within the limits of statistical fluctuations; they become different from zero only for $T > T_c$, and rise at most to the value $1/2$.⁽⁷⁾ For very high temperatures we expect that the damage decreases, because at $T = \infty$, $\bar{D}(G) = \bar{D}(M) = D(0) = 1/N$, as can be seen from Eqs. (7)–(9) and from corresponding relations for the dynamics M (not presented here). Close to T_c it is difficult to make a very precise statement because the large statistical fluctuations are responsible for numerical uncertainties.

(c) This agreement between the dynamics G and M is not valid when a magnetic field is applied (Fig. 2). The field decreases the damage; in particular, the temperature at which the damage becomes visibly different from zero is greater than T_c , and this effect becomes stronger with the field (see ref. 9 for a more extensive study of the effects of a magnetic field in the Ising model submitted to Glauber dynamics). It is impossible to say just from the numerical data if there is a threshold temperature above which the damages differ and below which they are the same or if in fact the damages differ over the whole range of temperatures to varying degree.

To study the damages produced by the dynamics HB and \tilde{M} , we have used another initial condition, namely $\sigma_i^A(0) = -\sigma_i^B(0)$ ($i = 1, \dots, N$), that is,

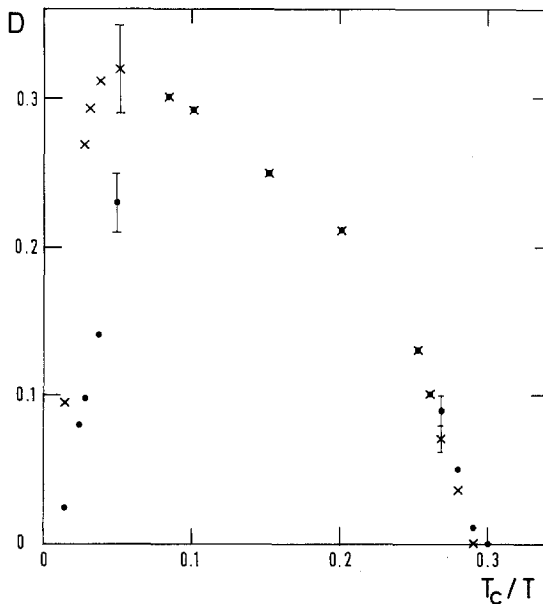


Fig. 2. The same as Fig. 1, but for $H/J = 1$.

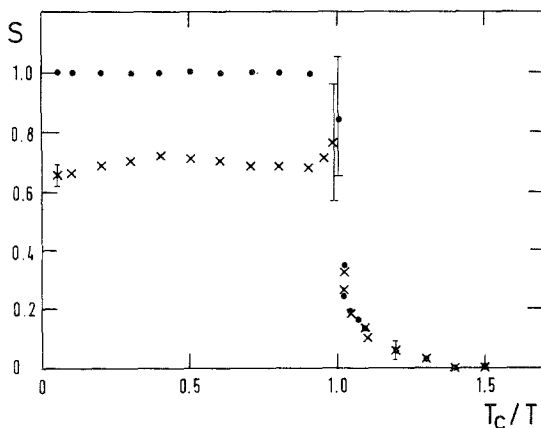


Fig. 3. The same as Fig. 1, for (●) heat-bath and (×) Metropolis \tilde{M} dynamics, with the initial condition $D(0)=1$. Here $S=1-D$ (fraction of sites at the same state in copies A and B).

$D(0)=1$. In this case, the damages $\bar{D}(G)$ and $\bar{D}(M)$ are constant for all temperatures [$\bar{D}(G)=\bar{D}(M)=D(0)=1$] and $\bar{D}(\tilde{M}) \approx \bar{D}(HB)$ (within the statistical error bars) only for $T \leq T_c$.

For high temperatures ($T > T_c$), we have found that $\bar{D}(HB) \approx 0$ and that $\bar{D}(\tilde{M})$ is nearly constant (≈ 0.30) for a large interval ($0.05 \leq T_c/T \leq 1$) of temperatures. In Fig. 3, we present $\bar{S} = 1 - \bar{D}$ as function of T_c/T for initial condition $D(0)=1$. At $T = \infty$, we expect $\bar{D}(HB) = 0$ and $\bar{D}(\tilde{M}) = D(0) = 1$, as can be shown from Eqs. (4) and (15)–(16). Note in Fig. 3 the large error bars in the critical region.

5. CONCLUSIONS

We have investigated the behavior of damage spreading in the Ising model submitted to four different dynamics: heat bath, Glauber, and two types of Metropolis (M and \tilde{M}).

By analyzing the probabilities for damage to appear, we have shown that in the G and M dynamics, the damage spreading is enhanced as compared to HB and \tilde{M} dynamics, respectively.

A two-dimensional numerical study suggests that for the damage spreading problem, at zero magnetic field, the dynamics G and M give the same result within statistical error bars. A similar agreement between the dynamics HB and \tilde{M} was observed (for zero magnetic field and low temperatures: $T < T_c$).

We remark that for the G and M dynamics, the value of $\sigma_i(t+1)$ strongly depends on the value of $\sigma_i(t)$, as opposed to HB and \tilde{M} dynamics.

This aspect may be crucial for the classification of other microscopic dynamics.

It might be interesting to investigate the three-dimensional case, where, for the damage spreading in Glauber dynamics, there is numerical evidence^(8,9) that the special temperature at which the damage becomes nonzero (at zero magnetic field) is not T_c , but very close ($\approx 0.96T_c$). In this case, a study investigating if the similarities and differences between the dynamics (G, M) and (HB, \tilde{M}) that we have observed in two dimensions are still valid would be welcome.

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